Lect23-0412b Invariant

Wednesday, April 13, 2016 9:47 AM

Homeomorphic or not

To prove $X = Y$ (meaning homeomorphic)

basically construct $f : X \longrightarrow Y$

$$S^1 \times (-\infty, \infty) \longrightarrow \mathbb{R}^2 \setminus \{0\}$$

$$(e^{i\theta}, t) \longmapsto (e^{+t}\cos\theta,\ e^{+t}\sin\theta)$$

But, to prove $X \neq Y$

cannot check all possible mappings

① $(\mathbb{R}, \text{standard}) \neq (S^1, \text{standard})$

why?

Answer. non-compact compact

② $([0,1], \text{standard}) \neq (S^1, \text{standard})$

why?

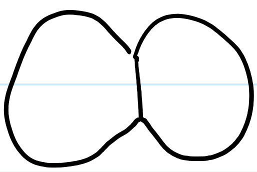
$\exists\, x_0 \in X$, $X \setminus \{x_0\}$ disconnected

$\forall\, y_0 \in Y$

$Y \setminus \{y_0\}$ connected

Exercise

$X \neq Y$

Exercise.  $\neq S^1$

Topological Property

Wednesday, April 13, 2016 9:47 AM

General Principle.

Find a topological property P, i.e.,
if X satisfies P then its homeomorphic image also satisfies P

① P: compactness

② P: $\exists\, x_0 \in X$ such that $X \setminus \{x_0\}$ is disconnected

In another form

① Define $k(X) = \begin{cases} 1 & \text{if } X \text{ is compact} \\ -1 & \text{if } X \text{ is non-compact} \end{cases}$

Fact. $X = Y \implies k(X) = k(Y)$

② Define $c(X) = \#$ of connected component

Clearly, $X = Y \implies c(X) = c(Y)$

But $c([0,1]) = 1 = c(S^1)$ no conclusion

Let $s(X) = \sup\{c(X \setminus \{x\}) : x \in X\}$

$$s([0,1]) = 2 \neq s(S^1) = 1$$

Fact. $X = Y \implies s(X) = s(Y)$

Topological Invariant

Function $\{\text{Topological spaces}\} \xrightarrow{\iota} \{\text{Mathematical objects}\}$

numbers, matrices
polynomials
vector spaces, etc.

satisfying
$X = Y \implies \iota(X) = \iota(Y)$

Euler Characteristic

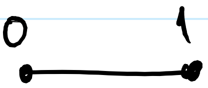
Wednesday, April 13, 2016 10:17 AM

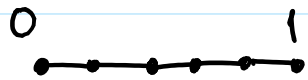
Euler Characteristic

$$\left\{\begin{matrix}\text{Topological}\\ \text{spaces}\end{matrix}\right\} \xrightarrow{\ \chi\ } \mathbb{Z}$$

$$\chi(X) \overset{\text{Roughly}}{=\!=} \begin{cases} V-E & \text{if } X \text{ is 1-dim} \\ V-E+F & \text{if } X \text{ is 2-dim} \\ \sum_{k=0}^{n} (-1)^k b_k & \text{if } X \text{ is } n\text{-dim} \end{cases}$$

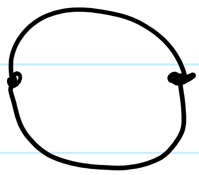
* $X = [0,1]$

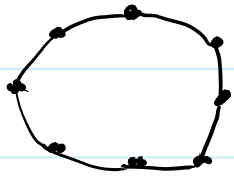


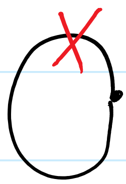


$\chi = 2-1 = 1$ or $6-5=1$

* $X = S^1$

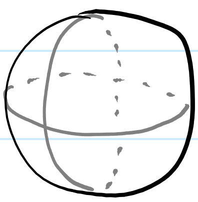






$\chi = 2-2=0$ or $8-8=0$

* $X = S^2$



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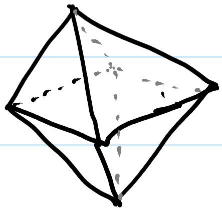

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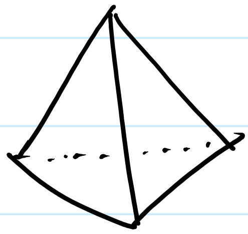
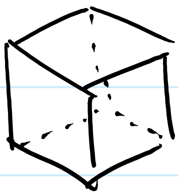
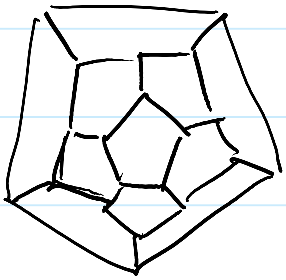


$\chi = 6-12+8=2$ or $4-6+4=2$



$\chi = 8-12+6=2$



Dodecahedron $= 20-30+12 = 2$

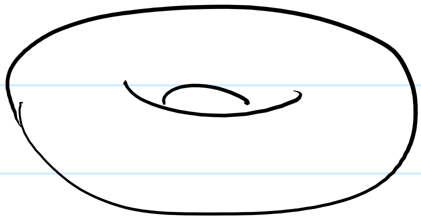
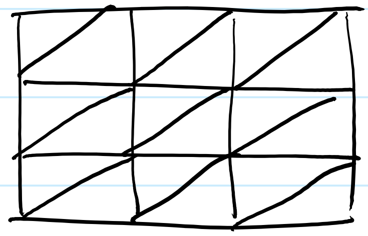
Icosahedron $= 12-30+20 = 2$

* Explore $\chi(\mathbb{R}) = 1$, $\chi(\mathbb{R}^2) = 1$, $\chi(\mathbb{R}^n) = 1$

Torus

Wednesday, April 13, 2016 10:48 AM

Torus $\mathbb{T}^2$

 $=$ 

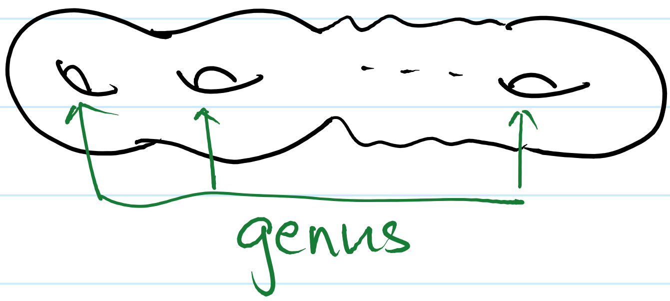
$$\chi(\mathbb{T}^2) = 9 - 27 + 18 = 0$$

$$\mathbb{T}^2 = S^1 \times S^1$$

$$\chi(\mathbb{T}^2) = \chi(S^1) \cdot \chi(S^1)$$

There is algebraic relation on topological invariants

Compact orientable surfaces



$$\chi = 2 - 2g$$

Compact surfaces

$$\chi(\mathbb{P}^2) = 1, \quad \chi(\text{Klein}) = -1$$

Fact. Let X, Y be compact surfaces.

$$X = Y \iff \chi(X) = \chi(Y)$$

Homotopy

Wednesday, April 13, 2016 3:17 PM

Continuous Change

Let X, Y be spaces. Two continuous maps $f, g : X \longrightarrow Y$ are **homotopic** if

$\exists$ continuous $H : X \times [0,1] \longrightarrow Y$, call **homotopy**

such that
$$\left.\begin{aligned} H(x,0) &= f(x) \\ H(x,1) &= g(x) \end{aligned}\right\} \quad \forall\ x \in X$$

Notation. $f \simeq g$ or $f \overset{H}{\simeq} g$

At every fixed t, we have

$$h_t : X \longrightarrow Y, \quad h_t(x) = H(x,t)$$

In other words, we have a family of mappings h_t, where $h_0 = f$, $h_1 = g$ and the family changes continuously in t.

Particular Example. $f = id : X \longrightarrow X$, $g : X \longrightarrow X$

Then $f \simeq g$ means a continuous variation from every point fixed (id) to $g(x)$.

Theorem. Homotopy is an equivalence relation.

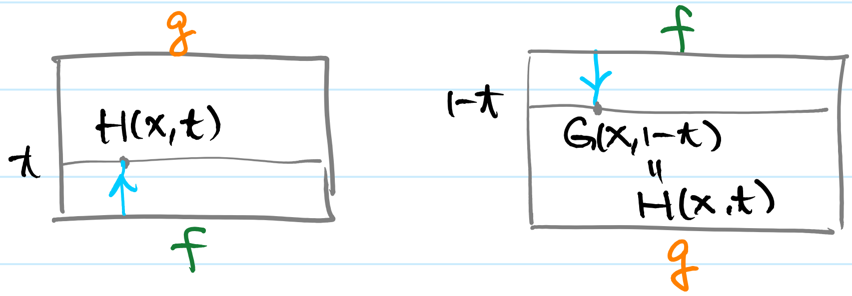
Reflexive. Obviously $f \simeq f$, $H(x,t) = f(x)$. $\forall\ t$

Equiv relation

Wednesday, April 13, 2016 3:46 PM

Symmetric Let $f \overset{H}{\simeq} g$, Then $g \overset{G}{\simeq} f$

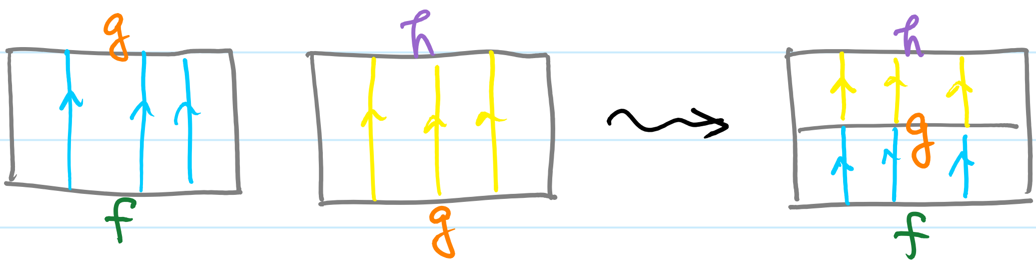
where $G(x,t) = H(x,1-t)$



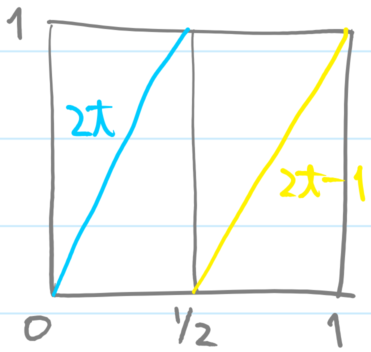
Transitive Let $f \overset{F}{\simeq} g$, $g \overset{G}{\simeq} h$ Then

$\exists\ H: X\times[0,1] \longrightarrow Y$, $f \overset{H}{\simeq} h$

$$H(x,t) = \begin{cases} F(x, 2t) & t \in [0,\tfrac{1}{2}] \\ G(x, 2t-1) & t \in [\tfrac{1}{2}, 1] \end{cases}$$



Adjust the clock



Conclusion

Let $C(X,Y)$ = the set of continuous maps $X \to Y$

$[X,Y] = C(X,Y)/\simeq$

$\uparrow$ set of homotopy classes